Bounds for polynomials with applications

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Communicated at the meeting of May 27, 1978

ABSTRACT

For a polynomial having a non-constant upper bound on an interval, we derive upper bounds valid outside of that interval. Several applications are given. The paper arose from a desire to have a simpler proof of a result of one of us and to extend it to the complex plane.

1. THE MAIN THEOREM

In the sequel, $Q_m(z)$ will denote a polynomial of degree m and with complex coefficients. In Theorem 1 it is assumed that

$$(1.1)\qquad |Q_m(x)| \leqslant C\Omega(x) \text{ for } x \in [-1, +1].$$

Here, C denotes a positive constant and further

$$(1.2)\qquad \Omega(z) = \prod_{k=1}^{N} |z-c_k|^{-s_k}.$$

The s_k are real constants and the c_k are complex constants.

For $k=1, \ldots, N$, define w_k as the complex constant such that

$$(1.3)\qquad c_k = \tfrac{1}{2}(w_k + w_k^{-1});\quad 0 < |w_k| \leqslant 1.$$

In fact, w_k is unique and satisfies $|w_k| < 1$ provided $c_k \notin [-1, +1]$. Other-

* This research was supported in part by the National Science Foundation.

wise, we have $c_k = \cos t_k$ with t_k real and we will choose $w_k = e^{it_k}$ or $w_k = ^{-it_k}$ in an arbitrary but fixed manner.

If c_k is real such that $c_k < -1$ then $-1 < w_k < 0$. More precisely, if $0 < a < 1$ then

$$(1.4) \qquad c_k = -(1+a^2)/(1-a^2) \text{ implies } w_k = -(1-a)/(1+a).$$

In a similar way, if z is a complex number then $w = w(z)$ is defined by

$$(1.5) \qquad z = \tfrac{1}{2}(w + w^{-1}); \quad 0 < |w| \leqslant 1.$$

Hence, $w = z \pm (z^2-1)^{1/2}$. We see from (1.3) and (1.5) that

$$(1.6) \qquad z - c_k = -(2w_k)^{-1}(1 - w_k w)(1 - w_k w^{-1}).$$

If $|w| = 1$ then $w^{-1} = \bar{w}$ so that

$$(1.7) \qquad |z - c_k| = |(2w_k)^{-1}(1 - w_k w)(1 - \bar{w}_k w)|, \textit{ provided } |w| = 1.$$

THEOREM 1. *Suppose that* (1.1) *holds for a polynomial* $Q_m(z)$ *of degree* m. *Then*

$$(1.8) \qquad |Q_m(z) \leqslant C|w|^{-m} \prod_{k=1}^{N} |(2w_k)^{-1}(1 - w_k w)(1 - \bar{w}_k w)|^{-s_k}$$

holds for all complex z. *An equivalent inequality is*

$$(1.9) \qquad |Q_m(z)| \leqslant C\,\Omega(z)|w|^{-m} \prod_{k=1}^{N} \left| \frac{w^{-1}(w - w_k)}{1 - \bar{w}_k w} \right|^{s_k}.$$

PROOF. That (1.8) and (1.9) are equivalent follows immediately from (1.2) and (1.6). In proving (1.8), one may assume that $C = 1$. Consider the pair of analytic functions

$$f(w) = w^m Q_m(\tfrac{1}{2}(w + w^{-1}))$$

and

$$g(w) = \prod_{k=1}^{N} \{(2|w_k|)^{-1}(1 - w_k w)(1 - \bar{w}_k w)\}^{s_k}.$$

Here, $f(w)$ is an entire function, in fact, a polynomial of degree $2m$. Since $|w_k| \leqslant 1$, the function $g(w)$ is analytic for $|w| \leqslant 1$, except that $|w_k| = 1$ leads to the singularities w_k and w_k^{-1} on $|w| = 1$. One can make g unique by requiring that $g(w)$ be real and positive for small real values w.

Observe that (1.8) (with $C = 1$) is equivalent to $|f(w)g(w)| \leqslant 1$ for $|w| \leqslant 1$. Hence, it suffices to show that $|f(w)g(w)| \leqslant 1$ for $|w| = 1$. And the latter follows immediately from (1.1), (1.2) and (1.7).

As a simple special case, suppose that

$$(1.10) \qquad |Q_m(x)| \leqslant 1 \text{ for } -1 \leqslant x \leqslant 1.$$

It follows from (1.8) with $C=1$ and $N=0$ that

$$(1.11) \qquad |Q_m(\tfrac{1}{2}(w+w^{-1}))| \leqslant |w|^{-m} \text{ when } |w| \leqslant 1.$$

Equivalently,

$$(1.12) \qquad |Q_m(\cos t)| \leqslant e^{n|\mathrm{Im}(t)|},$$

for all complex t. This is a well-known inequality due to Bernstein, see [2], p. 42. The upper bound in (1.11) is constant on each ellipse in the complex z-plane with foci -1 and $+1$.

REMARK. An alternative proof of Theorem 1 is as follows. The subharmonic function $h(w) = \log|f(w)|$ on $|w| \leqslant 1$ satisfies $|h(w)| \leqslant \phi(w)$ for $|w|=1$, where

$$\phi(w) = -\sum_{k=1}^{N} s_k \log|z-c_k|.$$

Hence, $|h(w)| \leqslant P(w)$ for $|w| \leqslant 1$, where $P(w)$ denotes the Poisson integral of the function $\phi(w)$ on $|w|=1$. Each of the N terms in $P(w)$ can be exactly computed employing a differentiation with respect to the parameter c_k. In this way one easily obtains (1.8). (In [3] a similar Poisson integral was used but only asymptotically evaluated.) This proof is applicable in certain cases when $\Omega(x)$ in (1.1) is not an analytic function.

2. AN APPLICATION WITH $N=1$

In the sequel, $0<a<1$, s and $C>0$ denote real constants.

THEOREM 2. *If a polynomial Q_m of degree m satisfies*

$$(2.1) \qquad |Q_m(x)| \leqslant Cx^{-s} \text{ for } 0<a^2 \leqslant x \leqslant 1$$

then we have for all complex values z that

$$(2.2) \qquad |Q_m(z)| \leqslant C|w|^{-m} \cdot \left|\frac{1+a}{2} + \frac{1-a}{2}w\right|^{-2s},$$

where w is defined by

$$(2.3) \qquad z = \frac{1+a^2}{2} + \frac{1-a^2}{2} \cdot \frac{w+w^{-1}}{2}; \quad |w| \leqslant 1.$$

An equivalent upper bound is

$$(2.4) \qquad |Q_m(z)| \leqslant C|z|^{-s}|w|^{-m} \cdot \left|\frac{(1+a)+(1-a)w^{-1}}{(1+a)+(1-a)w}\right|^{s}.$$

PROOF. Let us introduce

$$x = \frac{1-a^2}{2}x' + \frac{1-a^2}{2}; \quad Q_m(x) = \tilde{Q}_m(x').$$

It follows from (2.1) that

$$(\tilde{Q}_m('x)| \leqslant C'|x'-c'|^{-s} \text{ when } -1 \leqslant x \leqslant +1,$$

where

$$C' = C\left(\frac{1-a^2}{2}\right)^{-s}; \quad c' = -(1+a^2)/(1-a^2).$$

Applying (1.8) with $N=1$, we find that

(2.5) $$|\tilde{Q}_m(\tfrac{1}{2}(w+w^{-1}))| \leqslant C'|w|^{-m}|(2w_1)^{-1}(1-w_1w)^2|^{-s} \text{ for } |w| \leqslant 1.$$

Here, w_1 is defined by (1.3) with $c_1 = c'$. We see from (1.4) that w_1 is a real number, in fact $w_1 = -(1-a)/(1+a)$. Consequently, (2.5) reduces to (2.2) with w as defined by (2.3). Next observe that (2.3) implies

$$z = \left[\frac{1+a}{2} + \frac{1-a}{2}w\right] \cdot \left[\frac{1+a}{2} + \frac{1-a}{2}w^{-1}\right],$$

showing that (2.2) and (2.4) are equivalent.

REMARK. If $z=x$ is real with $a^2 \leqslant x \leqslant 1$ then $|w|=1$ so that the bound (2.4) coincides with the assumed inequality $|Q_m(x)| \leqslant Cx^{-s}$.

COROLLARY. If the polynomial Q_m of degree m satisfies (2.1) with $s \geqslant 0$ then

(2.6) $$|Q_m(x)| \leqslant Cx^{-s} \exp\left[2ma/(1-a^2)\right] \text{ for all } 0 < x \leqslant 1.$$

In particular, if $a = a_m = 0(1/m)$ then

(2.7) $$|x^s Q_m(x)| \leqslant CM \text{ for } 0 \leqslant x \leqslant 1,$$

(as long as a_m is bounded away from 1). Here M is a constant independent of m and x.

PROOF. If $a^2 \leqslant x \leqslant 1$ then (2.6) follows from (2.1). If $0 < x \leqslant a^2$ then (2.6) follows from (2.4). For, if $z=0$ then $w = -(1-a)/(1+a)$, (compare (1.4)), and if $z = a^2$ then $w=-1$. Thus, if $z=x$ satisfies $0 \leqslant x \leqslant a^2$ then $w=-u$ with $(1-a)/(1+a) \leqslant u \leqslant 1$. Observe that $w^{-1} = -u^{-1} \leqslant -u = w$, while

$$|w|^{-1} = u^{-1} \leqslant (1+a)/(1-a) \leqslant \exp\left[2a/(1-a^2)\right].$$

In many applications, one knows that $|Q_m(x)| \leqslant Cx^{-s}$ for all $0 < x \leqslant 1$. Then, for given z, the problem arises to choose the parameter $0 < a < 1$ in (2.3) or (2.4) in an optimal way. For example, if $z=0$ then $w = -(1-a)/(1+a)$ and (2.2) yields that

$$|Q_m(0)| \leqslant C(1+a)^{m+2s}(1-a)^{-m}(2a)^{-2s}.$$

If $s \leqslant 0$ then the best choice would be $a=0$. Let us assume that instead

$s \geqslant 0$. Then the best choice would be $a = \theta$, where

(2.8) $\quad \theta = s/(m+s)$.

This choice yields that

$$|Q_m(0)| \leqslant C\left(1+\frac{2s}{m}\right)^m\left(1+\frac{m}{2s}\right)^{2s}.$$

More generally, one has the following result. Here, we assume that

$$(2.9) \qquad H_m(z) = C|w|^{-m}\left|\frac{1+\theta}{2}+\frac{1-\theta}{2}w\right|^{-2s},$$

with θ as in (2.8), and that w is defined by (2.3) with $a=\theta$.

THEOREM 3. *Let $Q_m(x)$ be a polynomial of degree m satisfying*

$$(2.10) \qquad |Q_m(x)| \leqslant Cx^{-s} \text{ whenever } \left[\frac{s}{m+s}\right]^2 \leqslant x \leqslant 1,$$

where $s \geqslant 0$. Then

$$(2.11) \qquad |Q_m(x)| \leqslant H_m(x) \leqslant H_m(x_0) \text{ whenever } x_0 \leqslant x \leqslant 1,$$

(*where x_0 can be negative*). *In particular,*

$$(2.12) \qquad |Q_m(x)| \leqslant H_m(0) = C\left(1+\frac{2s}{m}\right)^m\left(1+\frac{m}{2s}\right)^{2s} \text{ whenever } 0 \leqslant x \leqslant 1.$$

PROOF. Apply Theorem 2 with $a=\theta$ and θ as in (2.8), thus, (2.1) is equivalent to (2.10). It follows from (2.2) that $|Q_m(z)| \leqslant H_m(z)$ for all complex z. If $z=0$ then $w=-(1-\theta)/(1+\theta)$ so that $H_m(0)$ is of the form (2.12). It suffices to show that $H_m(x)$ is a decreasing function on $(-\infty, +1]$.

From the Remark following Theorem 2, one has $H_m(x) = Cx^{-s}$ when $\theta^2 \leqslant x \leqslant 1$, which is certainly a decreasing function. Taking the derivative with respect to w, one easily verifies that $H_m(z)$ is an increasing function of $w \in [-1, 0)$ and thus a decreasing function of $z \in (-\infty, \theta^2]$.

REMARK 1. Suppose for instance that (2.10) holds with $s = 1/2$. Then (2.12) shows that

$$(2.13) \qquad |Q_m(x)| \leqslant C\left(1+\frac{1}{m}\right)^m (m+1) \text{ for } 0 \leqslant x \leqslant 1.$$

A slightly sharper result can be deduced from a theorem of Schur, see [2] p. 41. According to this theorem, $|Q_m(x)| \leqslant Cx^{-1/2}$ for $0 < x \leqslant 1$ implies that

$$(2.14) \qquad |Q_m(x)| \leqslant C(2m+1) \text{ for } 0 \leqslant x \leqslant 1.$$

The latter constant cannot be improved, as follows by choosing

$$Q_m(x) = T'_{2m+1}(\sqrt{1-x}),$$

with T_{2m+1} as the (odd) Tchebycheff polynomial of degree $2m+1$.

REMARK 2. If s is a positive integer then a result analogous to (2.12) can be derived as follows. Consider the polynomial $P_n(x)=x^sQ_m(x)$ of degree $n=s+m$. From the mean value theorem, we have $P_n(x)=Gx^s/s!$, where G is a point in the convex hull of $\{P_n^{(s)}(x'): 0\leqslant x'\leqslant x\}$. Since $|P_n(x)|\leqslant C$ for $\theta^2\leqslant x\leqslant 1$, from (2.10), the well-known Markov inequality (see [1]) yields an upper bound on $|P_n^{(s)}(x')|$. In this way one obtains an upper bound on $|Q_m(x)|=|P_n(x)x^{-s}|$ analogous to (2.12). One can show that it is always weaker than (2.12).

REMARK 3. Let $Q_m(x)$ be a polynomial of degree m satisfying

$$(2.15)\qquad |Q_m(x)|\leqslant C(1-x^2)^{-s} \text{ whenever } x^2\leqslant 1-\left[\frac{2s}{m+2s}\right]^2,$$

with C and s as nonnegative constants. Then

$$(2.16)\qquad |Q_m(x)|\leqslant C\left(1+\frac{4s}{m}\right)^{m/2}\left(1+\frac{m}{4s}\right)^{2s} \text{ whenever } -1\leqslant x\leqslant +1.$$

PROOF. The proof is analogous to that of (2.12). One starts off by applying (1.8) with $N=2$ to the polynomial

$$\tilde{Q}_m(x')=Q_m((1-\theta^2)^{1/2}x'), \text{ where } \theta=2s/(m+2s).$$

We omit the details.

3. BOUNDS FOR INCOMPLETE POLYNOMIALS

Let $\{P_n(x)\}_{n=1}^{\infty}$ be a sequence of polynomials such that P_n is of degree n and satisfies

$$(3.1)\qquad |P_n(x)|\leqslant 1 \text{ for } 0\leqslant x\leqslant 1.$$

Suppose further that P_n has $x=0$ as a zero of order $s(n)$ and that

$$(3.2)\qquad s(n)/n\to\theta \text{ as } n\to\infty.$$

Here and below, θ is a fixed constant with $0<\theta<1$.

It was shown by one of us [3] that these properties imply that

$$\lim_{n\to\infty} P_n(x)=0, \textit{ uniformly for } 0\leqslant x\leqslant\theta^2-\varepsilon,$$

this for each $\varepsilon>0$. The following is an extension of this result to the entire complex plane. We will even replace condition (3.1) by

$$(3.3)\qquad |P_n(x)|\leqslant 1 \text{ for } \theta^2\leqslant x\leqslant 1.$$

THEOREM 4. *Let $\Gamma=\Gamma(\theta)$ denote the bounded and open set defined by*

$$(3.4)\qquad z=\frac{1+\theta^2}{2}+\frac{1-\theta^2}{2}\cdot\frac{w+w^{-1}}{2},$$

where w satisfies both $|w| \leqslant 1$ *and*

$$(3.5)\qquad |w|^{-1}\cdot\left|\frac{(1-\theta)+(1+\theta)w}{(1+\theta)+(1-\theta)w}\right|^{\theta}<1.$$

If the polynomials $P_n(x)$ *of degree* n *satisfy* (3.2) *and* (3.3) *then*

$$(3.6)\qquad \lim_{n\to\infty} P_n(z)=0 \text{ for each } z\in\Gamma(\theta).$$

The latter convergence is uniform and exponentially fast for each compact subset of $\Gamma(\theta)$. *Similarly for each sequence of derivatives* $\{P_n^{(k)}(z)\}_{n=1}^{\infty}$, *($k$ fixed).*

PROOF. Let $m=m(n)=n-s(n)$, (thus, $m(n)/n\to 1-\theta$ as $n\to\infty$), and let $\theta(n)=s(n)/n$. One can write

$$(3.7)\qquad P_n(x)=x^{s(n)}Q_{m(n)}(x),$$

where Q_m is a polynomial of degree m. By (3.3), Q_m satisfies (2.1) with $C=1$ and $a=\theta$. Applying (2.4), one finds that

$$|P_n(z)|\leqslant|w|^{-m(n)}\cdot\left|\frac{(1+\theta)+(1-\theta)w^{-1}}{(1+\theta)+(1-\theta)w}\right|^{s(n)}.$$

Here, w is defined as in (3.4). An equivalent form is

$$(3.8)\qquad |P_n(z)|^{1/n}\leqslant|w|^{-1}\cdot\left|\frac{(1-\theta)+(1+\theta)w}{(1+\theta)+(1-\theta)w}\right|^{\theta(n)}.$$

The stated assertions are now an immediate consequence of (3.2) and (3.8). The derivative $P_n^{(k)}(z)$ is easily handled by using its Cauchy integral representation.

REMARK 1. The open set $\Gamma(\theta)$ in the complex z-plane intersects the real axis in an interval $(-\phi(\theta), \theta^2)$ about the origin. Here,

$$\phi(\theta)=\frac{1-\theta^2}{2}\cdot\frac{u+u^{-1}}{2}-\frac{1+\theta^2}{2},$$

where the number $0<u<(1-\theta)/(1+\theta)$ satisfies

$$\{(1-\theta)-(1+\theta)u\}/\{(1+\theta)-(1-\theta)u\}=u^{1/\theta}.$$

For instance, $\phi(2/3)=1/2$; $\phi(1/2)=1/8$; $\phi(1/3)=-\frac{1}{3}+\frac{2}{9}\sqrt{3}=.051567$.

REMARK 2. For the case where z is real, a different and independent proof of (3.8) was given by Saff and Varga [5]. They also established many related results. In particular, they proved that Theorem 4 is best possible in the following sense. Let z_0 be real with $z_0\notin\Gamma(\theta)$, that is, either $z_0\leqslant-\phi(\theta)$ or $z_0\geqslant\theta^2$. Then one can find a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ satisfying (3.1) and (3.2) and such that $\{P_n(z_0)\}_{n=1}^{\infty}$ is not a null-sequence. Professors

Saff and Varga kindly informed us that they have also succeeded in extending some of their results to the complex case.

REMARK 3. Suppose that condition (3.3) is replaced by

$$(3.9)\qquad |P_n(x)| \leqslant 1 \text{ for } \theta \leqslant |x| \leqslant 1, \quad (x \text{ real}).$$

In this situation (3.2) implies that

$$(3.10)\qquad \lim_{n\to\infty} P_n(x) = 0 \text{ for } -\theta < x < +\theta.$$

For the proof, one simply applies Theorem 5 to the polynomials

$$R_{n_1}(x') = (P_n(x) + P_n(-x))/2; \; R_{n_2}(x') = (P_n(x) - P_n(-x))/(2x)$$

of degrees $n_1 = [n/2]$ and $n_2 = [(n-1)/2]$, respectively, in terms of $x' = x^2$. These satisfy

$$|R_{n_1}(x')| \leqslant 1, \; |R_{n_2}(x')| \leqslant \theta^{-2} \text{ when } \theta^2 \leqslant x' \leqslant 1.$$

Analogous results hold for complex values z. Considering the polynomials $P_n(x) = R_{n/2}(x^2)$ and using the above mentioned result of Saff and Varga [5], it follows that the constant θ in (3.10) cannot be replaced by any larger number.

4. APPROXIMATION BY POLYNOMIALS: DIRECT THEOREMS

As one application of our inequalities, we give a simple proof of a theorem of Teljakovskii [6]. Since this is meant only as an illustration, we restrict ourselves to a special form of his theorem, namely, Theorem 5 below.

Let $p = 0, 1, \dots$ be an integer, let $0 < \alpha \leqslant 1$ and put $q = p + \alpha$. By $C^q[-1, +1]$ we denote the class of all functions f on $[-1, +1]$ having a p-th derivative $f^{(p)}$ which satisfies the Lipschitz condition

$$(4.1)\qquad |f^{(p)}(x) - f^{(p)}(y)| \leqslant |x-y|^\alpha \text{ if } x, y \in [-1, +1].$$

THEOREM 5. *For each function $f \in C^q[-1, +1]$ there exists a sequence of polynomials P_n of degree n such that, for $n \geqslant p+1$,*

$$(4.2)\qquad |f(x) - P_n(x)| \leqslant M\left(\frac{\sqrt{1-x^2}}{n}\right)^q \quad \text{if } -1 \leqslant x \leqslant +1.$$

Here the constant M depends only on q.

LEMMA 1. *Let the polynomial P_n of degree n satisfy*

$$(4.3)\qquad |P_n(x)| \leqslant C(1-x^2)^b \text{ if } |x| \leqslant 1 - n^{-2},$$

(x real). Here, C and b are positive constants. Let r be the smallest integer $\geqslant b$ and suppose that P_n has both -1 and $+1$ as a zero of order at least r

(thus $n \geqslant 2r \geqslant 2$). Then

$$(4.4) \qquad |P_n(x)| \leqslant CD(1-x^2)^b \textit{ for all } -1 \leqslant x \leqslant +1.$$

Here, D is a constant depending only on b. One may take $D = 2^b e^2$.

PROOF OF LEMMA 1. We can write $P_n(1-x) = x^r Q_m(x)$ with Q_m as a polynomial of degree $m = n - r$. From (4.3), we have that

$$|Q_m(x)| \leqslant Cx^{-r}[x(2-x)]^b \leqslant C2^b x^{-s},$$

(with $s = r - b \geqslant 0$), as long as $|1-x| \leqslant 1 - n^{-2}$, in particular, when $n^{-2} \leqslant x \leqslant 1$. We conclude from (2.6) with $a = n^{-1}$ that, for $0 \leqslant x \leqslant 1$,

$$|x^{-b}P_n(1-x)| = |x^s Q_m(x)| \leqslant C2^b \exp\left[\frac{2m}{n}(1-n^{-2})^{-1}\right] \leqslant C2^b e^2.$$

Here, we used that $m \leqslant n-1$ and $n \geqslant 2$. Thus, for $0 \leqslant x \leqslant 1$,

$$|P_n(x)| \leqslant C2^b e^2 (1-x)^b \leqslant C2^b e^2 (1-x^2)^b.$$

A similar proof holding for $-1 \leqslant x \leqslant 0$, this yields (4.4).

We shall also need the following result due to Trigub [7]; for other proofs, see [4] and [6]. Here and below,

$$(4.5) \qquad \Delta_n(x) = \max\{n^{-2}, n^{-1}\sqrt{1-x^2}\}.$$

LEMMA 2. *To each $f \in C^q[-1, +1]$ there corresponds a sequence of polynomials P_n of degree n such that, for $-1 \leqslant x \leqslant +1$,*

$$(4.6) \qquad |f^{(j)}(x) - P_n^{(j)}(x)| \leqslant K\Delta_n(x)^{q-j} \textit{ for } j = 0, 1, \ldots, p.$$

Here the constant K depends only on q.

PROOF OF THEOREM 5. We will prove a little more. Namely, let $q > 0$ and let r be an integer such that $q/2 \leqslant r \leqslant 1 + q/2$. Let further f be a function on $[-1, +1]$ having a derivative $f^{(r-1)}$ such that one can find a sequence P_n of polynomials of degree n satisfying

$$(4.7) \qquad |f^{(j)}(x) - P_n^{(j)}(x)| \leqslant \Delta_n(x)^{q-j} \text{ for } j = 0, 1, \ldots, r-1$$

and $-1 \leqslant x \leqslant +1$.

We claim that there exists a sequence $\tilde{P}_n$ of polynomials of degree n such that, for $n \geqslant 2r-1$ and $-1 \leqslant x \leqslant +1$,

$$(4.8) \qquad |f^{(j)}(x) - \tilde{P}_n^{(j)}(x)| \leqslant M(n^{-1}\sqrt{1-x^2})^{q-j} \text{ when } 0 \leqslant j \leqslant 2r-q,$$

(which includes $j = 0$ and possibly $j = 1, 2$). Here M denotes a constant depending only on q.

If $f \in C^q[-1, +1]$ then Lemma 2 implies (4.7). Hence, (4.8) holds with $j = 0$ which is precisely assertion (4.2).

Let f be as in (4.7). Let Q be the polynomial of degree $\leqslant 2r-1$ which interpolates f at ± 1 in such a way that

$$Q^{(j)}(-1)=f^{(j)}(-1) \text{ and } Q^{(j)}(+1)=f^{(j)}(+1) \text{ for } j=0, 1, \ldots, r-1.$$

Subtracting Q from both f and P_n, one may assume that f satisfies both (4.7) and

$$\text{(4.9)} \qquad f^{(j)}(-1)=f^{(j)}(+1)=0 \text{ for } j=0, 1, \ldots, r-1.$$

Applying (4.7) with $x=\pm 1$ and using (4.9), one has that

$$\text{(4.10)} \qquad |P_n^{(k)}(\pm 1)|\leqslant(1/n^2)^{q-k} \text{ for } k=0, 1, \ldots, r-1.$$

Next, let R_n denote the unique polynomial of degree $\leqslant 2r-1$ such that

$$\text{(4.11)} \qquad R_n^{(j)}(-1)=P_n^{(j)}(-1); \ R_n^{(j)}(+1)=P_n^{(j)}(+1), \text{ for } j=0, 1, \ldots, r-1.$$

We claim that, for $-1\leqslant x\leqslant +1$,

$$\text{(4.12)} \qquad |R_n^{(j)}(x)|\leqslant A\Delta_n(x)^{q-j} \text{ if } 0\leqslant j\leqslant q.$$

Here, A denotes a constant depending only on q. Namely,

$$\text{(4.13)} \qquad R_n(x)=\sum_{k=1}^{r-1} P_n^{(k)}(-1)p_k(x)+\sum_{k=0}^{r-1} P_n^{(k)}(+1)q_k(x),$$

with the p_k and q_k as the fundamental polynomials of Hermite interpolation. For example, $p_k^{(j)}(-1)=0$ if $0\leqslant j\leqslant r-1$ and $j\neq k$, while $p_k^{(k)}(-1)=1$. Further, $p_k^{(j)}(+1)=0$ for $0\leqslant j\leqslant r-1$. Hence, $p_k(x)$ is divisible by $(1+x)^k$ $(1-x)^r$ and thus by $(1-x^2)^k$. Therefore,

$$|p_k^{(j)}(x)|\leqslant \text{Const. } (1-x^2)^{(k-j)_+} \text{ if } -1\leqslant x\leqslant +1.$$

Here $a_+=\max(a, 0)$.

Let $0\leqslant k\leqslant r-1$; $0\leqslant j\leqslant q$ and $-1\leqslant x\leqslant +1$. Using (4.10) and (4.5), one obtains that

$$\begin{aligned}|P_n^{(k)}(-1)p_k^{(j)}(x)| &\leqslant \text{Const. } (1/n^2)^{q-k}\cdot(1-x^2)^{(k-j)_+}\\ &=\text{Const. } (1/n^2)^{q-k-(k-j)_+}(n^{-1}\sqrt{1-x^2})^{2(k-j)_+}\leqslant \text{Const. } \Delta_n(x)^{q-j}.\end{aligned}$$

Note that the latter inequality is obvious when $k\leqslant j\leqslant q$ in which case $(k-j)_+=0$. In the remaining case $0\leqslant j<k$, one also needs that $q-k-(k-j)_+\geqslant 0$, equivalently, $q-2k+j\geqslant 0$. This is true because $k\leqslant r-1\leqslant q/2$. A similar estimate holding for $P_n^{(k)}(+1)q_k^{(j)}(x)$, we see that (4.13) implies (4.12).

Let P_n denote the polynomial $\tilde{P}_n=P_n-R_n$. It is of degree n as soon as $n\geqslant 2r-1$. From (4.11), one has that

$$\text{(4.14)} \qquad \tilde{P}_n^{(j)}(-1)=\tilde{P}_n^{(j)}(+1)=0 \text{ for } j=0, 1, \ldots, r-1.$$

Moreover, from (4.7) and (4.12), for $-1\leqslant x\leqslant +1$,

$$\text{(4.15)} \qquad |f^{(j)}(x)-\tilde{P}_n^{(j)}(x)|\leqslant B\Delta_n(x)^{q-j} \text{ for } j=0, 1, \ldots, r-1.$$

Here $B=A+1$.

Let $0\leqslant j\leqslant r-1$ be fixed and consider the polynomial

$$(4.16)\qquad Q_{2N}(x)=\tilde{P}_{2N}^{(j)}(x)-\tilde{P}_{N}^{(j)}(x)$$

of degree $\leqslant 2N-j\leqslant 2N$ when $2N\geqslant 2r-1$. We have from (4.15) that

$$(4.17)\qquad |Q_{2N}(x)|\leqslant 2B\Delta_N(x)^{q-j} \text{ if } -1\leqslant x\leqslant +1.$$

If $|x|\leqslant 1-(2N)^{-2}$ then $(1-x^2)^{1/2}\geqslant(1-|x|)^{1/2}\geqslant(2N)^{-1}$, hence, $N^{-2}\leqslant(2/N)(1-x^2)^{1/2}$. Therefore, (4.17) yields that

$$(4.18)\qquad |Q_{2N}(x)|\leqslant 2B\left[\frac{2}{N}(1-x^2)^{1/2}\right]^{q-j} \text{ if } |x|\leqslant 1-(2N)^{-2}.$$

We now want to apply Lemma 1 with $b=(q-j)/2$. But then it is required that Q_{2N} have both -1 and $+1$ as a zero of order $\geqslant b$. This requires that

$$(4.19)\qquad Q_{2N}^{(k)}(-1)=Q_{2N}^{(k)}(+1)=0 \text{ whenever } 0\leqslant k<(q-j)/2.$$

From (4.14) and (4.16), a sufficient condition for (4.19) is that $k<(q-j)/2$ imply $j+k\leqslant r-1$ (k an integer); equivalently, that $j+k\geqslant r$ imply $k\geqslant(q-j)/2$. Taking $k=r-j$, this leads to the condition that $j\leqslant 2r-q$.

Assuming $j\leqslant 2r-q$, we have from (4.18), (4.19) and Lemma 1 that

$$|Q_{2N}(x)|\leqslant 2BD\left(\frac{2}{N}\right)^{q-j}(1-x^2)^{(q-j)/2}\leqslant C(N^{-1}\sqrt{1-x^2})^{q-j},$$

where $C=2^{q+1}BD$.

Finally, from (4.15) and (4.16),

$$|f^{(j)}-\tilde{P}_n^{(j)}|=\lim_i|\tilde{P}_{2^in}^{(j)}-\tilde{P}_n^{(j)}|\leqslant \sum_{i=1}^{\infty}|Q_{2^in}|,$$

yielding that

$$|f^{(j)}(x)-\tilde{P}_n^{(j)}(x)|\leqslant \sum_{i=1}^{\infty} C\{(2^{i-1}n)^{-1}\sqrt{1-x^2}\}^{q-j},$$

whenever $0\leqslant j\leqslant 2r-q$. This establishes (4.8).

5. APPROXIMATION BY POLYNOMIALS: INVERSE THEOREMS

For continuous functions f on $[0, 1]$ we define $\|f\|=\max\{|f(x)|: 0\leqslant x\leqslant 1\}$. We will show that, in a rough sense, a function $f\in C[0, 1]$ tends to inherit the zeros of a sequence $\{P_n\}$ of approximating polynomials as soon as $\|P_n-f\|$ tends to zero sufficiently fast. Or, to put it differently, if the degree of approximation is good enough then the polynomials P_n of best approximation cannot have zeros of order much higher than the function f itself. For simplicity, we restrict ourselves to the zero behavior at $x=0$.

In the sequel, $f\in C[0, 1]$ will be fixed and such that $\|f\|=1$. For s as a positive integer with $s<n$, let us define

$$(5.1)\qquad E_n(f, s)=E_n(s)=\inf_{P_n}\|P_n-f\|.$$

Here, P_n ranges over all polynomials of degree $\leqslant n$ which have $x=0$ as a zero of order $\geqslant s$.

Letting $P_n=0$, we see that $E_n(s)\leqslant\|f\|=1$. We shall denote by P_n the polynomial which achieves the infimum in (5.1), (it always exists). Then

$$\|P_n\|=\|f+(P_n-f)\|\leqslant\|f\|+E_n(s)\leqslant 2\|f\|=2.$$

Letting $P_n(x)=x^sQ_m(x)$, with Q_m as a polynomial of degree $m=n-s$, it follows from assertion (2.12) of Theorem 3 that

$$(5.2)\qquad |P_n(x)|\leqslant 2A_n(s)x^s \text{ if } 0\leqslant x\leqslant 1.$$

Here and below, the constant $A_n(s)$ is defined by

$$(5.3)\qquad A_n(s)=(n+s)^{n+s}/[(n-s)^{n-s}(2s)^{2s}].$$

A useful bound for $A_n(s)$ is

$$(5.4)\qquad A_n(s)\leqslant\left(\frac{ne}{2s}\right)^{2s}.$$

This can be seen by letting $s/n=t$ and

$$(1+t)\log(1+t)-(1-t)\log(1-t)=2t-\phi(t).$$

One has $\phi(t)\geqslant 0$ since $\phi(0)=0$ and $\phi'(t)=-\log(1-t^2)>0$, for $0<t<1$.

REMARK. Unless $x\in[0, 1]$ is close to 0, one could improve (5.2) by using (2.2). An inequality analogous to (5.2) (but weaker) could also be derived from the Markov inequalities, compare Remark 2 at the end of Section 2.

THEOREM 6. *For all* $0<s<n$ *one has*

$$(5.5)\qquad |f(x)|\leqslant E_n(s)+2A_n(s)x^s,\quad 0\leqslant x\leqslant 1.$$

Here $A_n(s)$ *is given by* (5.3)

PROOF. This follows from (5.2) and the inequality $|f(x)|\leqslant E_n(s)+|P_n(x)|$.

COROLLARY. Suppose that f has the lower bound

$$(5.6)\qquad |f(x)|\geqslant g(x) \text{ for } 0\leqslant x\leqslant x_0,$$

where $0<x_0<1$. Then

$$(5.7)\qquad E_n(s)\geqslant \sup\left\{g(x)-2\left(\frac{ne}{2s}\right)^{2s}\cdot x^s:\ 0\leqslant x\leqslant x_0\right\}.$$

The following is an application of the Corollary.

THEOREM 7. *Suppose that f satisfies*

(5.8) $$\liminf_{x \to 0} |f(x)x^{-r}| > 0,$$

where r is a positive constant. Then

(5.9) $$E_n(f, s) \geqslant M(s/n)^{2rs/(s-r)} \geqslant M s^{2r} n^{-2rs/(s-r)},$$

as soon as $0 < s < n$ are integers which satisfy $s > r$ and $n \geqslant cs$. Here, M and c denote positive constants depending only on f and r.

PROOF. There exist positive constants C and $x_0 < 1$ such that $|f(x)| \geqslant Cx^r$ when $0 \leqslant x \leqslant x_0$. It follows from (5.7) that

$$E_n(s) \geqslant Cx^r - 2\left(\frac{ne}{2s}\right)^{2s} \cdot x^s \text{ for each } x \in [0, x_0].$$

Select x such that the last term equals $-Cx^r/2$. This happens when

$$x = \left[\frac{C}{4}\left(\frac{2s}{ne}\right)^{2s}\right]^{1/(s-r)}.$$

One easily verifies that this defines a value $x \in [0, x_0]$ as soon as $n \geqslant cs$, where c is a sufficiently large positive constant, depending only on C and x_0. With this choice of x, we find that

$$E_n(s) \geqslant \tfrac{1}{2}Cx^r = \frac{C}{2}\left[\frac{C}{4}\left(\frac{2s}{ne}\right)^{2s}\right]^{r/(s-r)}.$$

Here, $(C/4)^{r/(s-r)}$ and $(2/e)^{2rs/(s-r)}$ are bounded away from zero, since both have a positive limit as $s \to \infty$. This proves (5.9).

COROLLARY. If s is fixed and $s > r$ then (5.8) implies that

(5.10) $$\liminf_{n \to \infty} E_n(f, s)n^t > 0,$$

where $t > 0$ is defined by

(5.11) $t = 2rs/(s-r)$, that is, $\dfrac{1}{r} = \dfrac{1}{s} + \dfrac{2}{t}$.

For instance, if $f(x) = x^{1/2}$ then $E_n(f, 2) \geqslant \text{Const.}\, n^{-4/3}$; if $f(x) = x$ then $E_n(f, 2) \geqslant \text{Const.}\, n^{-4}$. For such functions, it would be nice to know the best possible value t in (5.10).

As a further consequence of Theorem 7, one has that

(5.12) $$\liminf_{x \to 0} |f(x)x^{-r}| = 0,$$

as soon as there exists a sequence $\{s(n)\}$ of positive integers $s(n) > r$ such that $s(n)/n \to 0$ and further

(5.13) $$\liminf_{n \to \infty} \{E_n(f, s(n)) \cdot (n/s)^{2rs/(s-r)}\} = 0.$$

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